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Stability of Frequency Feedback Receivers Under Steps in Input Frequency

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It is known that frequency feedback demodulators can show instability in their response to step changes (mistuning) in input frequency. This work reports on some mathematical analyses of this phenomenon as described by differential equations arising from simple IF and feedback filters in the demodulator. These equations are studied for local and global stability by geometric or phase-plane analysis, by means of Lyapunov functions, and by the topological Poincaré-Bendixson methods. A typical result is for the case of no feedback filter and one-pole baseband analog of the IF filter, and states in physical terms that if the mistuning is not too big, specifically if

$$|\,\textit{mistuning}\,| < (\textit{half-power IF bandwidth})(1 + \textit{feedback gain})$$

then solutions which are bounded away from zero amplitude approach the natural equilibrium point. Examples are given in which a sufficiently large mistuning makes the equilibrium point unstable.

I. INTRODUCTION

The frequency feedback (or frequency compression) demodulator for FM signals was proposed by J. G. Chaffee[1] in 1937. After some

twenty-five years, Chaffee's idea was found to have a particularly fitting application in the satellite communications experiments Echo[2] and Telstar,[3] in which there was a high premium on detecting a low-power wide-band FM signal in noise. Nevertheless, since its invention, little progress has been made in the mathematical analysis of this circuit. Approximate methods of analysis and synthesis have been proposed, and some of them experimentally verified as useful.[4,5] However, except for unpublished works by S. O. Rice and T. R. Williams, the nonlinear character of the circuit away from equilibrium positions has not been considered.

It is the aim of this paper to formulate briefly one of the problems arising in the analysis of the FM with feedback (FMFB) receiver, namely that of stability of its response to step changes in input frequency. We shall write equations describing this response and present results about local and global stability of solutions for simple cases.

II. CIRCUIT DESCRIPTION

The FMFB receiver has been extensively discussed in recent publications,[4,5] so only a brief description of it is included here. Roughly speaking, the receiver is a conventional FM demodulator, with a local oscillator whose frequency is controlled linearly by the output of the detector. The object of this control is to reduce the index of modulation at the output of the mixer, so as to be able to use a narrower IF filter than in a conventional FM receiver, and thus to eliminate some of the noise accompanying the input signal. The action of the circuit is to follow the slowly varying frequency of an FM wave while looking at it through a moving narrow frequency "window."

The circuit is closely related to the phase-locked oscillator, but it is distinguished from that device by having amplitude effects absent in the latter. Mathematically this distinction takes the form that in FMFB there is an amplitude variable for every phase variable, while in phase-lock these variables do not appear. Their presence critically affects and complicates circuit analysis: thus the simplest FMFB equation is in two dimensions, while the simplest phase-lock equation is the pendulum equation, in one. The FMFB receiver resembles the phase-locked oscillator in that both devices work by phase-locking onto an FM wave that varies slowly over a limited range; if this range is exceeded locking fails and oscillation can set in. This phenomenon is well-known in phase-locked oscillators; in FMFB receivers a similar behavior has been described by L. H. Enloe.[4] It is to this stability prob-

lem that we address ourselves, endeavoring an analytical study of the stability of simple differential equations describing the mistuning of the incoming signal away from the normal carrier frequency.

A typical result we prove states that if the mistuning ω_d is not too big, specifically, in physical terms, if (for a one-pole baseband analog of the IF filter)

$$|\omega_d| < \text{(half-power IF bandwidth)}(1 + \text{feedback gain}),$$

then, for the simplest receiver, solutions which are bounded away from zero amplitude approach the equilibrium or critical point. This and similar results are proved by using the Poincaré-Bendixson theory, or with the help of Lyapunov functions.

III. EQUATIONS FOR RECEIVER WITH IDEAL DETECTOR

We shall write equations for the FMFB receiver (see Fig. 1) under the assumption that it contains an ideal frequency detector. That is, we assume that if the signal leaving the IF filter is $a(t) \cos (\omega t + \theta(t))$, then the detector produces the output $\dot{\theta}(t)$. Let the mixer input be

$$x_c \cos \omega_1 t - x_s \sin \omega_1 t,$$

and let the mixer multiply this input by

$$2 \cos (\omega_2 t + \beta\varphi(t))$$

where β (in practice and here > 0) is the feedback gain, and $\dot{\varphi}$ is the feedback signal. It is assumed that the IF filter is tuned to the difference frequency $\omega = \omega_1 - \omega_2$, and can be represented by an impulse response

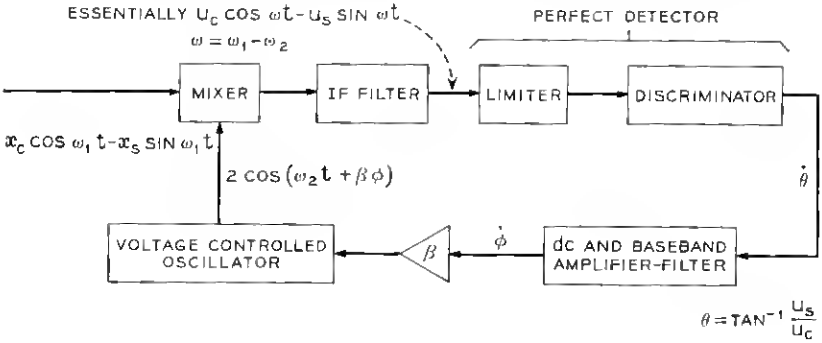

Fig. 1—FMFB receiver, block diagram.

of the form $2f(t)\cos\omega t$, with $f(\cdot)$ a baseband response such that $f(t) = 0$ for $t < 0$. The sum $(\omega_1 + \omega_2)$ components of the mixer are essentially removed by the IF filter, and will be ignored. The difference $(\omega_1 - \omega_2)$ components at the output of the mixer are

$$\cos\omega t\{x_c\cos\beta\varphi + x_s\sin\beta\varphi\} - \sin\omega t\{x_s\cos\beta\varphi - x_c\sin\beta\varphi\}.$$

The response of the IF filter to these components has the form

$$\cos\omega t\int_0^t f(t-u)\{x_c(u)\cos\beta\varphi(u) + x_s(u)\sin\beta\varphi(u)\}\,du$$

$$- \sin\omega t\int_0^t f(t-u)\{x_s(u)\cos\beta\varphi(u) - x_c(u)\sin\beta\varphi(u)\}\,du$$

$$+ \text{ terms around } 2\omega$$
$$+ \text{ terms representing initial conditions.}$$

We shall assume that the passband of $f(\cdot)$ is small compared to 2ω, so that the components around 2ω may be ignored as well.

To complete the loop equations we must indicate how the feedback $\varphi(\cdot)$ is determined from the output of the IF filter. We set, for $t \geq 0$

$$u_c(t) = r_c(t) + \int_0^t f(t-u)\{x_c(u)\cos\beta\varphi(u) + x_s(u)\sin\beta\varphi(u)\}\,du,$$

$$u_s(t) = r_s(t) + \int_0^t f(t-u)\{x_s(u)\cos\beta\varphi(u) - x_c(u)\sin\beta\varphi(u)\}\,du,$$

where $r_c(\cdot)$, $r_s(\cdot)$ represent the effects of initial conditions in the filter at $t = 0$. Exclusive of the carrier, the angle modulation of the IF output,

$$u_c\cos\omega t - u_s\sin\omega t,$$

is just

$$\theta = \tan^{-1}\frac{u_s}{u_c},$$

corresponding to an instantaneous frequency,

$$\dot\theta = \frac{u_c\dot u_s - u_s\dot u_c}{a^2},$$

where $a = (u_c^2 + u_s^2)^{1/2}$. This is the output of the ideal detector.

The feedback frequency $\dot\varphi(\cdot)$ controlling the voltage-controlled oscillator is obtained by filtering $\dot\theta(\cdot)$. Thus

$$\dot{\varphi}(t) = r(t) + \int_0^t k(t-u)\dot{\theta}(u)\,du, \qquad t \geq 0$$

where $k(\cdot)$ is the impulse response of the feedback filter, and $r(\cdot)$ represents the effect of initial conditions at $t = 0$.

IV. DIFFERENTIAL EQUATION FOR THE SIMPLEST CASE

When the baseband responses $f(\cdot)$ and $k(\cdot)$ correspond to filters with rational transfer functions, the integral equations for $u_c(\cdot)$ and $u_s(\cdot)$ can be turned into differential equations in a well-known way. In the simplest case, when there is no feedback filter and $f(\cdot)$ corresponds to a (one-pole no-zero) filter with transfer $\mu/(\lambda + s)$, we obtain the equations

$$\begin{aligned}
\dot{u}_c &= -\lambda u_c + \mu[x_c \cos \beta\theta + x_s \sin \beta\theta] \\
\dot{u}_s &= -\lambda u_s + \mu[x_s \cos \beta\theta - x_c \sin \beta\theta] \\
\dot{\theta} &= \mu \frac{u_c(x_s \cos \beta\theta - x_c \sin \beta\theta) - u_s(x_c \cos \beta\theta + x_s \sin \beta\theta)}{u_c^2 + u_s^2}.
\end{aligned}$$

The introduction of polar coordinates $u_c = a \cos \theta$, $u_s = a \sin \theta$ simplifies these equations to

$$\begin{aligned}
\dot{\theta} &= \frac{\mu}{a}(x_s \cos(\beta + 1)\theta - x_c \sin(\beta + 1)\theta) \\
\dot{a} &= -\lambda a + \mu(x_c \cos(\beta + 1)\theta + x_s \sin(\beta + 1)\theta).
\end{aligned} \tag{1}$$

We first consider the stability of equations (1) when the input to the demodulator consists of the carrier $\cos \omega_1 t$ alone, with no signal. In this case $x_c \equiv 1$, $x_s \equiv 0$, and the equations are

$$\begin{aligned}
\dot{\theta} &= -\frac{\mu}{a} \sin(\beta + 1)\theta \\
\dot{a} &= -\lambda a + \mu \cos(\beta + 1)\theta.
\end{aligned} \tag{2}$$

We recall that the critical points of a differential equation $\dot{x} = v(x)$ are the points x in the phase-space at which $v(x) = 0$. Those of the system (2) are then the points in the a, θ plane at which simultaneously $\dot{\theta} = \dot{a} = 0$, namely,

$$a = \frac{\mu}{\lambda}, \qquad \theta = \frac{2n\pi}{\beta + 1}, \qquad n \text{ an integer}.$$

Because of the periodic dependence of the right-hand side of (2) on θ,

it is possible and convenient to define $\zeta = (\beta + 1)\theta$, to write (2) as

$$\dot{\zeta} = -(\beta + 1)\frac{\mu}{a}\sin\zeta \qquad (3)$$

$$\dot{a} = -\lambda a + \mu\cos\zeta,$$

and to consider only principal values of ζ, and thus only the critical point $(\mu/\lambda,\ 0)$ in the plane specified by the polar coordinates $(a,\ \zeta)$.

Theorem 1: The equations (3) are globally asymptotically stable for all positive λ and μ; all solutions tend to the critical point μ/λ, 0 in an exponential manner; ζ is monotone, and a is either monotone or has one minimum.

Proof: We start with an heuristic direct analysis of the trajectories. Consider in Fig. 2 the circle C in the a, ζ plane defined by $\dot{a} = 0$, that is, $a = \mu/\lambda\cos\zeta$. With $x = a\cos\zeta$, $y = a\sin\zeta$ we shall examine the directions of the trajectories of (3) at points on C. The equation of C is

$$y = \left(\frac{\mu x}{\lambda} - x^2\right)^{\frac{1}{2}}.$$

Since C is the locus $\dot{a} = 0$, it is apparent that on C each trajectory (has a tangent that) is perpendicular to the radius from the origin.

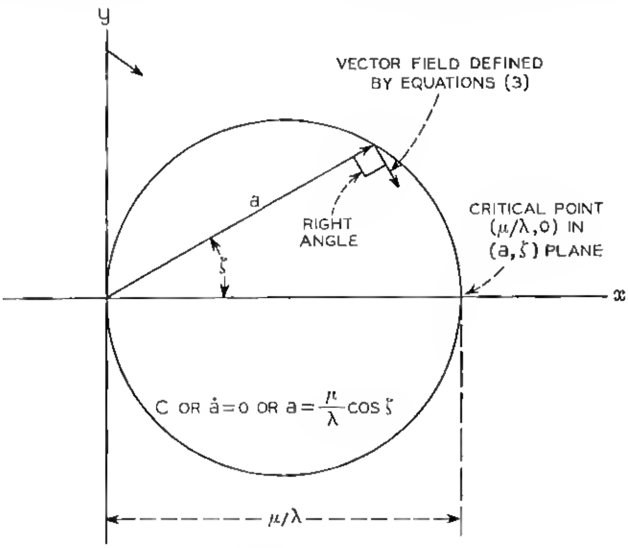

Fig. 2—Phase plane for no mistuning.

By symmetry about the x-axis, we can restrict attention to $y \geqq 0$. The slope of C at x, y is

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{\mu x}{y} - x^2\right)^{-\frac{1}{2}}\left(\frac{\mu}{\lambda} - 2x\right) = \frac{1}{2y}\left(\frac{\mu}{\lambda} - 2x\right),$$

while the slope of the line through x, y perpendicular to the radius from the origin is

$$\frac{y}{x - \frac{\mu}{\lambda}}.$$

Since for $0 < x < \mu/\lambda$,

$$\left(\frac{\mu}{\lambda} - 2x\right)\left(x - \frac{\mu}{\lambda}\right) < 2\left(\frac{\mu x}{\lambda} - x^2\right),$$

we find

$$\frac{\mu}{\lambda} - 2x > \frac{2y^2}{x - \frac{\mu}{\lambda}},$$

or since $y > 0$ and $0 < x < \mu/\lambda$

$$\frac{dy}{dx} = \frac{1}{2y}\left(\frac{\mu}{\lambda} - 2x\right) > \frac{y}{x - \frac{\mu}{\lambda}}.$$

Thus every trajectory is entering C on $\dot{a} = 0$ except at the origin and at the critical point. ζ is decreasing in $\zeta > 0$. If a trajectory ever crosses C it can never again recross it and must approach the critical point; in this case a has a single minimum. If a trajectory never crosses C, it must simply slip into the critical point, because then both a and ζ decrease and are bounded below.

These preliminaries lead us to define the Lyapunov function

$$V = \frac{1}{2}\left(\frac{\mu}{\lambda} - a\cos\zeta\right)^2 + \frac{1}{2}(a\sin\zeta)^2$$

$$= \frac{1}{2}\,(\text{distance from } a, \zeta \text{ to critical point})^2$$

Evidently $V \geqq 0$, and $V = 0$ only at μ/λ, 0. The rate of change of V along trajectories of (3) is

$$\dot{V} = a\dot{a} - \dot{a}\frac{\mu}{\lambda}\cos\zeta + a\frac{\mu}{\lambda}\dot{\zeta}\sin\zeta$$

$$= -\lambda\left(a - \frac{\mu}{\lambda}\cos\zeta\right)^2 - \mu^2\frac{\beta + 1}{\lambda}\sin^2\zeta$$

$$= -2\lambda V - \frac{\mu^2\beta}{\lambda}\sin^2\zeta < 0$$

except at the critical point, where $V = \dot{V} = 0$. It follows from Theorem II, p. 37 of Ref. 6, that the system (3) is globally asymptotically stable: all solutions tend exponentially to the critical point with reciprocal time constant 2λ. When $\lambda = \mu$, 2λ has the physical interpretation

$$2\lambda = 2 \times (\text{half-power IF bandwidth}).$$

V. MISTUNING IN THE SIMPLEST CASE

Let us assume that in equation (1) we have

$$x_s = \sin\omega_d t, \qquad x_c = \cos\omega_d t,$$

corresponding to the "mistuned" carrier input $\cos(\omega_1 + \omega_d)t$, or to the constant modulating signal ω_d. The equation (1) assumes the form

$$\dot{\theta} = \frac{\mu}{a}\sin(\omega_d t - (\beta + 1)\theta)$$

$$\dot{a} = -\lambda a + \mu\cos(\omega_d t - (\beta + 1)\theta),$$

or with $\zeta = \omega_d t - (\beta + 1)\theta$,

$$\dot{\zeta} = \omega_d - \frac{\mu(\beta + 1)}{a}\sin\zeta \tag{4}$$

$$\dot{a} = -\lambda a + \mu\cos\zeta.$$

The critical point of this system is determined by the conditions

$$a = \frac{\mu}{\lambda}\cos\zeta, \qquad \zeta = \tan^{-1}\frac{\omega_d}{\lambda(\beta + 1)} \tag{5}$$

It is important to note that because of the possibility of going to low amplitudes there always exist critical points, regardless of the value of ω_d. This situation is in sharp contrast with the phase-locked oscillator. For a filterless phase-locked oscillator the equation corresponding to (4) would be

$$\dot{\zeta} = \omega_d - \mu\sin\zeta,$$

which has no critical point if $\omega_d > \mu$. Thus in phase-lock there is usually a critical frequency deviation above which locking is impossi-

ble for lack of critical points, and below which it may or may not occur. In the FMFB receiver, though, the critical points always exist but, as we shall see later, they are not always stable.

We determine the stability of the critical point (5) by the standard method of linearization. The matrix

$$\begin{pmatrix} \dfrac{\partial}{\partial \zeta} \dot{\zeta} & \dfrac{\partial}{\partial a} \dot{\zeta} \\[2ex] \dfrac{\partial}{\partial \zeta} \dot{a} & \dfrac{\partial}{\partial a} \dot{a} \end{pmatrix} = \begin{pmatrix} -\dfrac{\mu(\beta+1)}{a} \cos \zeta & \dfrac{\mu(\beta+1)}{a^2} \sin \zeta \\[2ex] -\mu \sin \zeta & -\lambda \end{pmatrix}$$

of partial derivatives, evaluated at the critical point, is the matrix A appropriate for the linearized system. The determinant of $(sI\text{-}A)$ turns out to be

$$s^2 + s\lambda(\beta + 2) + \lambda^2(\beta + 1) + \frac{\omega_d^2}{\beta + 1},$$

with roots in the left half-plane. Hence the critical point is stable; in a neighborhood of it the trajectories approach it.

Because of the symmetry of the equations, there is no loss of generality in assuming, as we do henceforth, that $\omega_d < 0$. This convention is used in Figs. 3 and 4.

Although we have not proved it, it is natural (and we conjecture) that a separatrix lies between solutions which pass around the origin in the upper half of the a, ζ plane, and solutions which just miss the origin as they go past it in the lower half of the plane. Roughly speaking, the former pick up an extra 2π of phase before settling. This separatrix may even be a fan[7] of solutions each of which goes into the origin, although in all likelihood it consists of a single trajectory. This conclusion is supported by a heuristic low-amplitude analysis of equations (4) suggested by J. A. Morrison. He writes (4) as the single equation

$$\frac{d\zeta}{da} = \frac{a\omega_d - \mu(\beta + 1) \sin \zeta}{\mu a \cos \zeta - \lambda a^2},$$

and for $|\cos \zeta| \cong 1$ drops terms of order a^2, obtaining

$$\mu a \cos \zeta \frac{d\zeta}{da} = a\omega_d - \mu(\beta + 1) \sin \zeta = \mu a \frac{d}{da} \sin \zeta,$$

whence by integration from a_0 to a

$$\sin \zeta - \frac{\omega_d a}{\mu(\beta + 2)} = \left(\frac{a_0}{a}\right)^{\beta+1} \left(\sin \zeta_0 - \frac{\omega_d a_0}{\mu(\beta + 2)}\right).$$

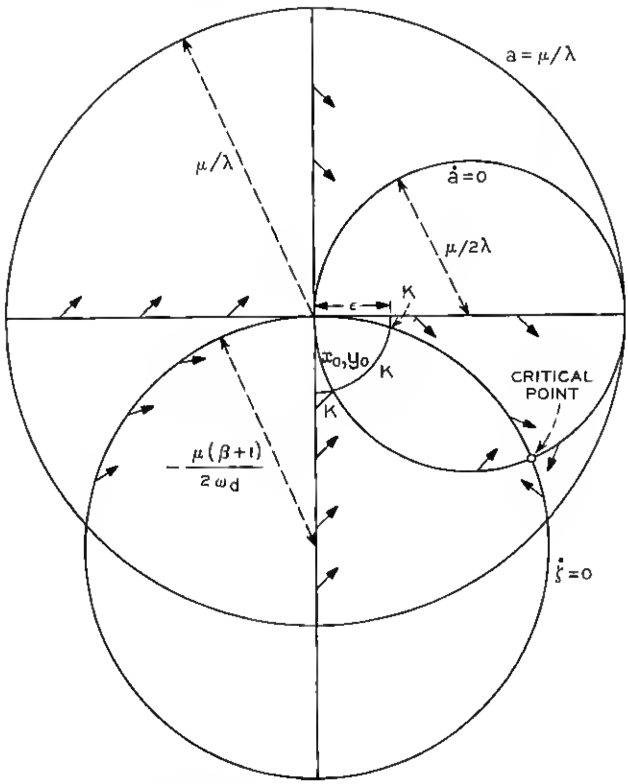

Fig. 3—Phase plane for mistuning.

This formula suggests that if a point (a_0, ζ_0), on the circle

$$a = \frac{\mu(\beta + 2)}{\omega_d} \sin \zeta$$

and near the origin, is on a trajectory then a nearby point on the circle is on the same trajectory. In other words, a trajectory going through the origin does so like the circle above, which is tangent to but outside the circle $\dot{\zeta} = 0$ with equation

$$a = \frac{\mu(\beta + 1)}{\omega_d} \sin \zeta.$$

To avoid difficulties we shall consider only trajectories which are bounded away from the origin.

We now address ourselves to the global stability of the mistuned equations (4). Since the system is two dimensional it is possible to try to use the topological Poincaré-Bendixson methods.[7] This is most conveniently done by passing to rectangular coordinates $x = a \cos \zeta$, $y = a \sin \zeta$ again, and calculating the divergence. We rewrite (4) as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\lambda x + \dfrac{\mu x^2}{x^2 + y^2} + \mu(\beta + 1) \dfrac{y^2}{x^2 + y^2} - \omega_d y \\ -\lambda y - \dfrac{\mu \beta x y}{x^2 + y^2} + \omega_d x \end{pmatrix}$$

$$= v(x, y)$$

We find

$$\frac{\partial \dot{x}}{\partial x} = -\lambda - 2\mu\beta \frac{xy^2}{(x^2 + y^2)^2}$$

$$\frac{\partial \dot{y}}{\partial y} = -\lambda - \mu\beta x \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\operatorname{div} v = -2\lambda - \frac{\mu\beta x}{x^2 + y^2}$$

$$= -2\lambda - \frac{\mu\beta}{a} \cos \zeta.$$

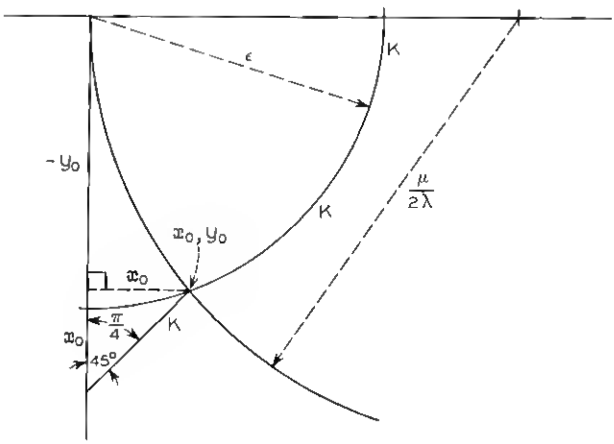

Fig. 4—Details of the curve K.

Lemma 1: In the portion of the fourth quadrant comprised by an arbitrary neighborhood of the origin there is always a curve K, joining the positive x-axis to the negative y-axis, such that on K trajectories of (4) cross K in the outward direction, i.e., out of the part cut off by K near the origin, and into the part separated by K from the origin. (See Fig. 3.)

Proof: Let K consist of the circle $a = \epsilon$ from the x-axis down to the point where this circle crosses $\dot{a} = 0$, i.e., until $\cos \zeta = \epsilon\lambda/\mu$. The Cartesian coordinates of this point are

$$x_0 = \frac{\lambda\epsilon^2}{\mu}, \qquad y_0 = -\frac{\epsilon}{\mu}\sqrt{\mu^2 - \epsilon^2\lambda^2}$$

From here let K continue to the y-axis at slope 1, i.e., let it consist of that portion of the line

$$y = x - \frac{\epsilon}{\mu}\sqrt{\mu^2 - \epsilon^2\lambda^2} - \frac{\lambda\epsilon^2}{\mu}$$

which is between its intercept $y_0 - x_0$ on the y-axis and the circle $\dot{a} = 0$.

Now on $a = \epsilon$ inside $\dot{a} = 0$ we have $\dot{a} > 0$, so on the circular part of K all trajectories are entering $a > \epsilon$, even at x_0, y_0. At x_0, y_0 the trajectory is actually tangent to $a = \epsilon$, but for small enough ϵ it is pointed in the direction of increasing x and so there too it must enter $a > \epsilon$.

On the linear part of K we want to verify that

$$\frac{dy}{dx} = \frac{-\lambda y + \omega_d x - \mu\beta \dfrac{xy}{x^2 + y^2}}{-\lambda x + \mu - \omega_d y + \mu\beta \dfrac{y^2}{x^2 + y^2}} < 1$$

if ϵ is small enough. This is true because on the linear part of K we have $x \to 0$, $y \to 0$, and

$$\cos \zeta = \frac{x}{\sqrt{x^2 + y^2}} \to 0$$

all monotonely and uniformly, as $\epsilon \to 0$; near $\epsilon = 0$ the denominator of dy/dx is close to $\mu(\beta + 1)$ for $(x, y) \in K$, so on the linear part of K the trajectories are moving in the direction of increasing x at a slope $dy/dx < 1$; hence they are crossing K in the direction of increasing amplitude. This proves the lemma.

Theorem 2: If $|\omega_d| < \lambda(\beta + 1)$, *then every trajectory of (4) that is bounded away from the origin approaches the critical point* $a = \mu/\lambda \cos \zeta$, $\zeta = \tan^{-1} \omega_d/\lambda(\beta + 1)$.

Proof: It suffices to consider only $\omega_d < 0$. All trajectories outside $a = \mu/\lambda$ have $\dot{a} < 0$, so it is enough to consider those starting inside, because the others get there eventually. Consider a path starting in $0 \leqq \zeta \leqq 3\pi/2$, $\dot{\zeta} \leqq 0$, $a \leqq \mu/\lambda$, and bounded away from the origin. Either it stays in this region forever, or it reaches the fourth quadrant, or else it moves into $\dot{\zeta} > 0$. If it stays there forever then there is a closed region, free of critical points and excluding the origin, in which it stays. By a result of Poincaré (Ref. 7, p. 232), this closed region also contains a closed path γ, of period say τ. Then because γ is closed, if $\zeta(0)$ is on γ,

$$\zeta(\tau) - \zeta(0) = \int_0^\tau \dot{\zeta}(t)\, dt = 0.$$

But this is impossible since $\dot{\zeta} < 0$ throughout the region in question.

In the third quadrant a trajectory can cross $\dot{\zeta} = 0$ only once. The argument just given also shows that no path bounded away from the origin can stay in the region $\pi \leqq \zeta \leqq 3\pi/2$, $\dot{\zeta} > 0$, $a \leqq \mu/\lambda$. Thus all paths starting in the region $a \leqq \mu/\lambda$ and bounded away from the origin reach the fourth quadrant.

The inequality in the hypothesis implies that the circle $\dot{\zeta} = 0$ intersects the circle $a = \mu/\lambda$. Away from the origin we have $\dot{\zeta} < 0$ for $x = 0$, $y > 0$ and $\dot{\zeta} > 0$ for $x = 0$, $y < 0$. On $0 < x \leqq \mu/\lambda$, $y = 0$ the trajectories enter the fourth quadrant intersected with $a \leqq \mu/\lambda$. Since a is nonincreasing on $a = \mu/\lambda$, it follows from Lemma 1 that there is a region R with these properties:

(i) R is closed.
(ii) $R \subseteq \{a \leqq \mu/\lambda\} \cap$ fourth quadrant.
(iii) $(\{a \leqq \mu/\lambda\} \cap$ fourth quadrant $- R)$ is in an arbitrarily small neighborhood of the origin.
(iv) R is entered by the path under consideration.
(v) R is invariant, i.e. maps into itself under the motion.

Indeed R can be chosen to be a 2-cell (homeomorph of a disc). We note that the divergence is negative throughout R. It follows from the criterium of Bendixson (Ref. 7, p. 238) that R contains no limit cycles nor even an oval going to and from a critical point. Since R contains only one critical point, it can contain no path-polygon. Thus two of the three alternatives in Bendixson's theorem (Ref. 7, p. 230) are ruled out, and all paths starting in R go to the critical point. Since we can associate a region like R with any trajectory bounded away from the origin, the theorem is proved.

We remark that when $\lambda = \mu$ the condition of the theorem can be rendered in physical terms as

$$|\text{mistuning}| \leqq (\text{half-power IF bandwidth})(1 + \text{feedback gain}).$$

We next show that a result similar to Theorem 2 can be obtained by a Lyapunov function argument.

Theorem 3: If

$$|\omega_d| < \frac{2\lambda(\beta + 1)^{\frac{3}{2}}}{\beta},$$

then every trajectory of (4) that is bounded away from the origin approaches the critical point $a = \mu/\lambda$, $\zeta = \tan^{-1} \omega_d/\lambda(\beta + 1)$.

Proof: Consider the scalar function V defined by

$$\begin{aligned} 2V &= (\lambda a - \mu \cos \zeta)^2 + (\beta + 1)^{-2}(a\omega_d - \mu(\beta + 1) \sin \zeta)^2 \\ &= \dot{a}^2 + (\beta + 1)^{-2}(a\dot{\zeta})^2. \end{aligned}$$

We find

$$\begin{aligned} \dot{V} = &-\lambda(\lambda a - \mu \cos \zeta)^2 \\ &+ \frac{\beta\omega_d}{(\beta + 1)^2}(\lambda a - \mu \cos \zeta)(a\omega_d - \mu(\beta + 1) \sin \zeta) \\ &- \frac{\lambda}{\beta + 1}(a\omega_d - \mu(\beta + 1) \sin \zeta)^2. \end{aligned}$$

$-\dot{V}$ is a quadratic form in $\dot{a}$ and $a\dot{\zeta}$ with determinant

$$\begin{vmatrix} \lambda & \dfrac{-\beta\omega_d}{2(\beta + 1)^2} \\ \dfrac{-\beta\omega_d}{2(\beta + 1)^2} & \dfrac{\lambda}{\beta + 1} \end{vmatrix}$$

which is positive whenever $|\omega_d| < 2\lambda(\beta + 1)^{3/2}/\beta$.

Consider now a trajectory bounded away from the origin. It is clearly bounded, so it has a positive limiting set Γ^+ which is invariant and to which it tends. There is a constant k such that the trajectory is entirely contained in a bounded subregion Ω of $\{V < k\}$. Hence $\Gamma^+ \subseteq \Omega$ and $\dot{V} = 0$ on Γ^+. Since a is bounded away from 0 on the trajectory, it follows that $\dot{a} = 0$ and $\dot{\zeta} = 0$ on Γ^+. Thus the trajectory tends to the critical point. (This is a variant of the argument for Theorem VI, p. 58

of Ref. 6.) Again, the condition of the theorem is that $|\omega_d|$ not be too big, viz.,

$$|\omega_d| < 2(\text{half-power IF bandwidth}) \times (\beta + 1)^{3/2}/\beta,$$

where β is the feedback gain.

VI. ONE-POLE FILTER IN THE FEEDBACK

After the filterless case considered so far, the next simplest model for the FMFB receiver would have one-pole, no-zero filters both as the baseband equivalent $f(\cdot)$ of the IF response, and as the response $k(\cdot)$ in the feedback. This is the simplest case that has appreciable practical import: the IF filter corresponds closely to the one-mesh design described by Giger and Chaffee (loc. cit., p. 1119 and Fig. 5, p. 1120); the feedback filter and the gain β are a rudimentary version of the dc-and-baseband amplifier sketched by these authors (loc. cit. pp. 1121–22.)

In this case the differential equations for the system are

$$\begin{aligned}
\dot{u}_c &= -\lambda u_c + \mu(x_c \cos \beta\varphi + x_s \sin \beta\varphi)\\
\dot{u}_s &= -\lambda \mu_s + \mu(x_s \cos \beta\varphi - x_c \sin \beta\varphi)\\
\dot{\theta} &= a^{-2}(u_c \dot{u}_s - u_s \dot{u}_c) = \frac{d}{dt} \tan^{-1} \frac{u_s}{u_c}\\
\dot{x} &= -\gamma x + \delta\dot{\theta}\\
\dot{\varphi} &= x
\end{aligned}$$

with $a = (u_c^2 + u_s^2)^{\frac{1}{2}}$ as before, and $\delta/(\gamma + s)$ the transfer function of the feedback filter. Upon setting $\xi = \theta + \beta\varphi$ these simplify to

$$\begin{aligned}
\dot{\xi} &= \beta x + \frac{\mu}{a}(x_s \cos \xi - x_c \sin \xi)\\
\dot{a} &= -\lambda a + \mu(x_c \cos \xi + x_s \sin \xi)\\
\dot{x} &= -\gamma x + \frac{\delta\mu}{a}(x_s \cos \xi - x_c \sin \xi).
\end{aligned}$$

When the modulating signal is a constant ω_d, then $x_s(t) = \sin \omega_d t$, $x_c(t) = \cos \omega_d t$, and with $\zeta(t) = \omega_d t - \xi(t)$ the equations become

$$\dot{\zeta} = \omega_d - \beta x - \frac{\mu}{a} \sin \zeta$$

$$\begin{aligned} \dot{a} &= -\lambda a + \mu \cos \zeta \\ \dot{x} &= -\gamma x + \frac{\delta\mu}{a} \sin \zeta. \end{aligned} \tag{6}$$

Let us note heuristically and physically that if $\delta = \gamma \to \infty$ then the feedback bandwidth goes to ∞ and we obtain the equations (4) of the simplest case.

We start with a study of the stability of the critical point ζ_0, a_0, x_0 defined by

$$\begin{aligned} \zeta_0 &= \tan^{-1} \frac{\omega_d}{\lambda\left(1 + \dfrac{\beta\delta}{\gamma}\right)} \\ a_0 &= \frac{\mu}{\lambda} \cos \zeta_0 \\ x_0 &= \frac{\delta\omega_d}{\gamma + \beta\delta}. \end{aligned} \tag{7}$$

The matrix $A = (\partial f_i/\partial x_j)$ of partial derivatives evaluated at the critical point is

$$\begin{array}{cc} & \begin{array}{ccc} \zeta & a & x \end{array} \\ \begin{array}{c} \dot{\zeta} \\ \dot{a} \\ \dot{x} \end{array} & \begin{pmatrix} -\lambda & \dfrac{\lambda^2}{\mu} \dfrac{\tan \zeta_0}{\cos \zeta_0} & -\beta \\ -\mu \sin \zeta_0 & -\lambda & 0 \\ \delta\lambda & \dfrac{\delta\lambda^2}{\mu} \dfrac{\tan \zeta_0}{\cos \zeta_0} & -\gamma \end{pmatrix} \end{array}.$$

The determinant of $(sI\text{-}A)$ is

$$\begin{aligned} (s + \lambda)((s + \lambda)(s + \gamma) + \beta\delta\lambda) &+ \delta^2\lambda^2\beta \tan^2 \zeta_0 + \lambda^2 \tan^2 \zeta_0(s + \gamma) \\ &= s^3 + (2\lambda + \gamma)s^2 + (\lambda\gamma + \beta\delta\lambda + \lambda^2 \tan^2 \zeta_0)s \\ &\quad + \lambda^2(\beta\delta + \tan^2 \zeta_0(\beta\delta + \gamma)) \\ &= s^3 + a_2 s^2 + a_1 s + a_0 . \end{aligned}$$

A necessary and sufficient condition for stability is that

$$a_1, a_2, a_0 > 0 \quad \text{and} \quad a_2 a_1 > a_0 .$$

The first three are clearly true, and the last amounts to

$$(2\lambda + \gamma)(2\lambda\gamma + \lambda^2 + \lambda\beta\delta) + \left(\frac{\gamma\omega_d}{\gamma + \beta\delta}\right)^2 > \lambda^2\beta\delta + \frac{\gamma^2\omega_d^2}{\gamma + \beta\delta}.$$

This is symmetric in $\pm\omega_d$, and is true for $|\omega_d|$ small enough. It becomes false for large $|\omega_d|$ if $2\lambda < \beta\delta$. If $\lambda = \mu$ and $\gamma = \delta$, then these numbers are the half-power bandwidths of the IF and feedback filter respectively, and we may say in physical terms that if

$$\beta = \text{feedback gain} < 2 \times \frac{\text{IF bandwidth}}{\text{feedback bandwidth}} = \frac{2\lambda}{\delta}$$

then a very large mistuning cannot affect the local stability of the system, but if β exceeds twice the ratio of IF to baseband widths then sufficiently large mistuning will make the system unstable. This result was first observed in an unpublished work (although with some errors) of T. R. Williams.

The global stability of the equations (6) for the case with a one-pole in the feedback is a far more difficult topic than the local. Naturally, as more complicated filters are assumed for the IF and the feedback, the dimension of the problem goes up, and the kind of geometric analysis we are using here becomes virtually impossible. In particular, the Poincaré-Bendixson theory used earlier is already unavailable in three dimensions, and also there seems to be no ready way to prove the boundedness of solutions. Nevertheless some information can be obtained from the construction of a Lyapunov function for the case of no mistuning; all attempts to extend the method to the case of mistuning have failed.

Theorem 4: If $\omega_d = 0$ (no mistuning) then every trajectory of (6) that is bounded away from the line $a = 0$ approaches the critical point given by (7).

Proof: Consider the scalar function V defined by

$$2V = \frac{\dot{a}^2}{\lambda} + \frac{a^2\dot{\zeta}^2}{\lambda + \gamma + \delta\beta} + \frac{\mu^2}{\lambda}\sin^2\zeta + \frac{\mu^2\cos^2\zeta}{\lambda + \gamma + \delta\beta}.$$

V is certainly positive along any trajectory satisfying the hypothesis. We find after a lot of elementary calculus that

$$\dot{V} = -(\dot{a})^2 - \frac{(\lambda + \gamma)a^2(\dot{\zeta})^2}{\lambda + \gamma + \delta\beta}.$$

Since the trajectory assumed in the theorem is bounded away from the line $a = 0$, it is easy to see from the equations (6) that it is bounded. Thus the positive limiting set of this trajectory is a nonempty, compact invariant set Γ^+, to which it tends. There is a constant k such that the

trajectory is eventually in a bounded subregion Ω of $\{V < k\}$. Thus $\Gamma^+ \subseteq$ and $\dot{V} = 0$ on Γ^+. Consider now the largest invariant subset M of $\{\dot{V} = 0\} \cap \Omega$. Clearly $\Gamma^+ \subseteq M$. Thus the trajectory tends to M. On M, $\dot{V} = 0$; hence since the trajectory is bounded away from $a = 0$, we see also that $\dot{a} = 0$ and $\dot{\zeta} = 0$ on M. Now the equations $\dot{a} = 0$, $\dot{\zeta} = 0$ define a spiral curve C on the cylinder $\lambda a = \mu \cos \zeta$ by the formula

$$\psi = -\frac{\lambda}{\beta} \tan \zeta,$$

and M is an invariant subregion of this curve, bounded away from $a = 0$ (which C is not). On C the vector field defined by the equations must either vanish, or else must point in the $+\psi$ direction, or else point in the $-\psi$ direction; this is because there can be no motion in the a, ζ plane on $\dot{a} = 0$, $\dot{\zeta} = 0$. If either of the second two alternatives holds at a point of C, that point cannot belong to M, because the trajectory through it would move off C and $M \subseteq$ C. Hence M consists of C-points at which the field vanishes, i.e., $M = \{\text{critical point}\}$. (Cf. Theorem VI, p. 58 of Ref. 6).

Try as we might (and we tried many Vs) we have not succeeded in proving a version of Theorem 4 in which there was mistuning. If the same V is used with $\omega_d \neq 0$ as was used in Theorem 4, then it is no longer true that $\dot{V} < 0$; thus the results we feel are there still elude proof.

VII. COMPENSATED ATTENUATOR IN FEEDBACK

In private communication, L. H. Enloe and B. R. Davis have suggested that probably the most important practical FMFB receiver is one having a single-pole (as the baseband equivalent of the) IF filter, and a single-pole—zero-feedback filter, i.e., one with transfer

$$\frac{s + a}{s + b}. \tag{8}$$

In the time domain this acts like a delta-function plus an exponential. From formula (6) we see that the right-hand side of the equation for $\dot{\zeta}$ can be thought of as the output of a filter whose response is a delta-function plus an exponential. This suggests that the analysis in Section 6 can be made to cover the filter transfer (8) as well as the one-pole, because the differential equations are such as naturally to supply the constant in (8) if it is not present.

Writing the transfer function as

$$\frac{s + a}{s + b} = 1 + \frac{c}{s + b}$$

with $c = a - b$, the differential equations for the system become

$$\dot{u}_c = -\lambda u_c + \mu(x_c \cos \beta\varphi + x_s \sin \beta\varphi)$$

$$\dot{u}_s = -\lambda u_s + \mu(x_s \cos \beta\varphi - x_c \sin \beta\varphi)$$

$$\dot{\theta} = a^{-2}(u_c\dot{u}_s - u_s\dot{u}_c) = \frac{d}{dt} \tan^{-1} \frac{u_s}{u_c}$$

$$\dot{\varphi} = \dot{\theta} + y$$

$$\dot{y} = -by + c\dot{\theta},$$

with $a = (u_c^2 + u_s^2)^{1/2}$ again. We note that

$$\dot{\theta} = \frac{\mu}{a} (x_s \cos (\beta\varphi + \theta) - x_c \sin (\beta\varphi + \theta));$$

now we set $\xi = \beta\varphi + \theta$ and simplify the equation to

$$\dot{\xi} = \beta y + \frac{(\beta + 1)\mu}{a} (x_s \cos \xi - x_c \sin \xi)$$

$$\dot{a} = -\lambda a + \mu(x_c \cos \xi + x_s \sin \xi)$$

$$\dot{y} = -by + \frac{c\mu}{a} (x_s \cos \xi - x_c \sin \xi).$$

With the modulating signal a constant ω_d, we have as for equation (6), $x_s(t) = \sin \omega_d t$, $x_c(t) = \cos \omega_d t$, and we can set $\zeta(t) = \omega_d t - \xi(t)$ to obtain the equations

$$\begin{aligned} \dot{\zeta} &= \omega_d - \beta y - \frac{(\beta + 1)\mu \sin \zeta}{a} \\ \dot{a} &= -\lambda a + \mu \cos \zeta \\ \dot{y} &= -by + \frac{c\mu}{a} \sin \zeta. \end{aligned} \tag{9}$$

These equations have the same form as (6) except that c replaces δ, b replaces γ, and there is an extra $(\beta + 1)$ coefficient in the sin term of $\dot{\zeta}$. The critical point is at

$$a_0 = \frac{\mu}{\lambda} \cos \zeta_0$$

$$\zeta_0 = \tan^{-1} \frac{\omega_d}{\lambda(\beta + 1)\left(1 + \frac{c\beta}{b}\right)}$$

$$y_0 = \frac{\lambda c}{b} (\beta + 1) \tan \zeta_0 .$$

The matrix of partial derivatives evaluated at the critical point is

$$\begin{array}{c|ccc} & \zeta & a & y \\ \hline \dot\zeta & -(\beta+1)\lambda & (\beta+1)\dfrac{\lambda^2}{\mu}\dfrac{\tan\zeta_0}{\cos\zeta_0} & -\beta \\ \dot a & -\mu\sin\zeta_0 & -\lambda & 0 \\ \dot y & c(\beta+1)\lambda & -c(\beta+1)\dfrac{\lambda^2}{\mu}\dfrac{\tan\zeta_0}{\cos\zeta_0} & -b \end{array}.$$

The appropriate polynomial is

$$\begin{aligned} s^3 &+ s^2(\lambda(\beta+1)+b+\lambda)+s(\lambda b(\beta+2) \\ &\quad + \lambda^2(\beta+1)(1+\tan^2\zeta_0)+c\lambda\beta(\beta+1)) \\ &\quad + b(\beta+1)\lambda^2\tan^2\zeta_0 + c\lambda^2\beta(\beta+1)+c(\beta+1)\beta\lambda^2\tan^2\zeta_0 \\ &= s^3+a_2s^2+a_1s+a_0\,. \end{aligned}$$

A necessary and sufficient condition for local stability is then that a_0, a_1, and a_2 all > 0, and that $a_2a_1 > a_0$. It is clear that the first three conditions are met whenever $c > 0$, i.e., $a > b$. The case $a = b$ is degenerate and reduces in dimension to the filterless case of Section 4. Also, a_2 is always positive. However, since

$$\tan\zeta_0 = \frac{\omega_d}{\lambda(\beta+1)\left(1+\dfrac{c\beta}{b}\right)}$$

we can write a_1 as

$$a_1 = \lambda b + \lambda^2(\beta+1)+\lambda(\beta+1)(b+c\beta)+\frac{|\,\omega_d\,|^2\,b^2}{(b+c\beta)^2(\beta+1)}.$$

If

$$b\left(\beta-1-\frac{1}{\beta+1}\right) > \lambda+\beta a$$

then the sum of the first three terms is negative, and $a_1 < 0$ for $|\omega_d|$ *sufficiently small.* Similarly

$$a_0 = c\lambda^2\beta(\beta+1)+\frac{|\,\omega_d\,|^2\,(b+c\beta)}{\left(1+\dfrac{c\beta}{b}\right)^2(\beta+1)^2}$$

which is negative for *any* ω_d if $b + c\beta < 0$, and is also negative for $|\omega_d|$ sufficiently small if $c < 0$. The case $c < 0$ is physically a bit strange, because it is equivalent to having positive feedback in one loop of the feedback path; thus it is not surprising that in this case there can be instability even for $\omega_d = 0$.

In the case $c > 0$ only the condition $a_2 a_1 > a_0$ is of concern and this is

$$(\lambda\beta + 2\lambda + b)\lambda b(\beta + 2) + \lambda^2(\beta + 1)(1 + \tan^2 \zeta_0) + c\lambda\beta(\beta + 1) > (b + \beta c)(\beta + 1)\lambda \tan^2 \zeta_0 + c\lambda^2\beta(\beta + 1),$$

which simplifies somewhat to

$$\lambda^2 b(\beta^2 + 7\beta + 9) + \lambda^3(\beta + 1)(\beta + 2) + c\lambda\beta(\beta + 1)(b + \lambda\beta + \lambda) + \lambda b^2(\beta + 2) > \frac{|\,\omega_d\,|^2}{(\beta + 1)^2\left(1 + \dfrac{c\beta}{b}\right)^2}(\lambda^2(\beta + 1)\beta c - \lambda^3(\beta + 1)(\beta + 2)).$$

Again, this is symmetric in $\pm\omega_d$, and is true for $|\omega_d|$ small enough. It becomes false for $|\omega_d|$ large if

$$\beta c > \lambda(\beta + 2).$$

We can think of the feedback path in this example as consisting of two parallel branches whose sum is added, one being an amplifier with gain, 1, the other being a single-pole filter S with dc gain 1 and (half-power) bandwidth b, in series with an amplifier A of gain c/b. We can then say in physical terms, assuming $\lambda = \mu$, that if

$$\beta < (\beta + 2) \times \frac{IF \text{ bandwidth}}{S \text{ bandwidth}} \times \text{gain of } A$$

then even a very large mistuning cannot affect local system stability, but if not, then a sufficiently large mistuning can. The fact that $(\beta + 2)$ appears as a factor on the right shows how much the zero in the compensating attenuator helps prevent instability, in agreement with what has been observed in practice by L. H. Enloe and B. R. Davis (private communication).

The global stability of the equations (9) may be studied by the same methods as were used in Section 6 for that of equations (6), but this topic is not pursued further here.

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